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Research paper

A proposal for a formula of absolute pole velocities between relative poles

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ABSTRACT

In this paper is proposed a vectorial equation that relates the absolute pole velocities of three moving rigid bodies with a planar motion of general type. From this equation, it is possible to obtain a relation between the pole velocities of the three mathematical points, related between them by the Aronhold-Kennedy Theorem. The formula allows the calculation of one of the pole velocities from the other two, being known the angular velocities and accelerations of the moving bodies. It is applicable regardless of whether the instantaneous centers (poles) are located on physical points on the linkage or not. Illustrative examples of the application of the formula on representative planar linkages are included. In the final section, is discussed a similar concept associating a mathematical point to the curvature centers of a point's path, so called centroma.

1. Introduction and approach to the problem

The instantaneous center (pole) of a rigid body with planar motion has the property of being a physical point that instantaneously (or even permanently) has no speed. Consequently, the planar motion of a rigid body can be studied as a sequence of differential rotations about an axis perpendicular to the plane containing the pole.

This point has a physical-mathematical duality. The physical point, which belongs to the moving body, is the one whose velocity is zero. But the mathematical point has the so called pole velocity, in the direction of the centrode tangent. In this paper, the velocities corresponding to linkage joints or physical points will be identified by the vector $\boldsymbol{v}$ while those corresponding to pole velocities will have the vector $\boldsymbol{u}$ associated.

The poles can be absolute if they have null velocity with respect to the frame, or relative having null velocity in a relative motion between two bodies. Authors like Hunt [1] have already investigated in the relations obtained when referring to the study of the pole in the relative motion. The present research is based on [2], where the planar motion bases and the motion of a mechanism are explained with the geometry that underlies behind.

All the examples analysed and the calculations done, have been simulated and verified using the GIM research and educational numerical software [3]. This software is being developed by the COMPMECH research group of the University of the Basque Country - UPV/EHU (www.ehu.es/compmech). Also in [4], a relationship for computing several angular velocities of rigid bodies can be found.

The curvature theory and the envelope theory are presented in detail in [5,6]. In references [6,7] the general form of Euler-Savary equation, together with the Aronhold theorem and Hartmanś construction are explained. The inflection circle

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Nomenclature

A_i	Geometric point A of moving body i.
P_{ij}	Instantaneous center (pole) of velocity of moving body i with respect to moving body j.
$\mathbf{u}_{ij}$	Absolute pole velocity of P_{ij}.
$\boldsymbol{\omega}_i$	Angular velocity of moving body i.
$\boldsymbol{\omega}_{i_{r_j}}$	Angular velocity of moving body i relative to moving body j.
$\rho_{f\mathbf{i}}$	Radius of curvature of the fixed centrode of moving body i.
$\rho_{m\mathbf{i}}$	Radius of curvature of the moving centrode of moving body i.
$\boldsymbol{\alpha}_i$	Angular acceleration of moving body i.
$\boldsymbol{\alpha}_{i_{r_j}}$	Angular acceleration of moving body i relative to moving body j.
$\boldsymbol{v}_C$	Velocity of physical point C.
O_A	Centroma of A (mathematical point associated to the centre of curvature of the path of point A).
$\mathbf{r}_{IJ}$	Position vector relating points I and J.

and the cuspidal circle are explained in [8] while in [9] is clearly presented the concept of instantaneous center and the relative velocity field. In [10,11] the kinematic analysis of complex mechanisms is presented.

In references [12,13] the concepts of relative angular velocity and acceleration are introduced and applied in representative examples. In [14], new formulas for the first and second time derivatives of 2×2 transforms based on the Cayley-Klein parameters are derived. Based on these, an extension to the computation of velocities and accelerations of the kinematic analysis proposed by Denavit [15] is presented.

Reference [16] investigates the instantaneous spatial higher pair to lower pair substitute-connection which is kinematically equivalent up to acceleration analysis for two smooth surfaces in point contact. In [17] using the contact kinematics equations of the enveloping curves, is shown how the theorem on coordinated centers is valid for a position in which the instantaneous relative angular velocity is zero [18]. This is possible since the approach does not make any reference to the polodes.

Fig. 1 shows the case of pure rolling motion between two disks being 1 the frame. The null velocities of the absolute poles P_{12}, P_{13}, the velocity of the relative pole P_{23} and the absolute pole velocities $\mathbf{u}_{12}$, $\mathbf{u}_{12}$ y $\mathbf{u}_{23}$ are depicted.

Being $\boldsymbol{\omega}_i$ the angular velocity of the moving body *i*, ρ_f and ρ_m the radius of curvature of the fixed and moving centrodes respectively and using the Euler-Savary formula, u_{12} and u_{13} are,

$$u_{12} = \frac{\omega_2}{\frac{1}{\rho_{f2}} - \frac{1}{\rho_{m2}}}; \quad u_{13} = \frac{\omega_3}{\frac{1}{\rho_{f3}} - \frac{1}{\rho_{m3}}} \tag{1}$$

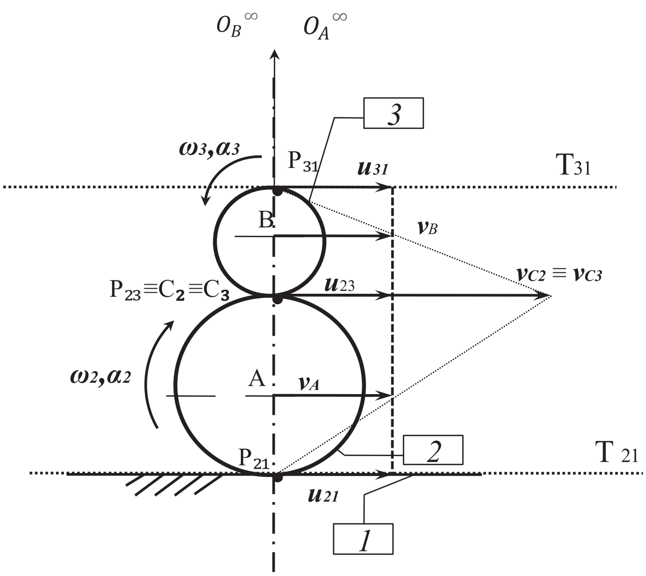

Fig. 1. Rolling motion between two disks.

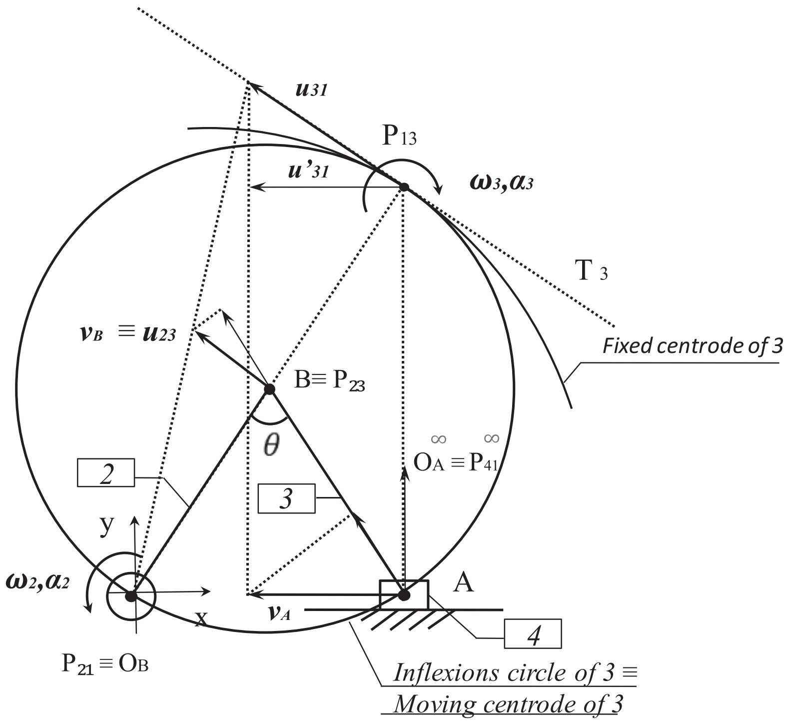

Fig. 2. RRRP linkage.

The approach of the present work is to obtain a formula for calculating, in a general case, the absolute pole velocity of the relative pole ($\mathbf{u}_{23}$ in Fig. 1) as a function of the pole velocities of the absolute poles ($\mathbf{u}_{12}$ and $\mathbf{u}_{13}$ in Fig. 1), the angular velocities ($\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$) and accelerations ($\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$) of the two moving rigid bodies.

Focusing on the kinematics of linkages, sometimes the relative pole between two bodies lies permanently on a physical point. This is the case of the revolute pairs. As it has been said before, this relative pole has the same absolute velocity belonging to each of the mechanism's links. This concept is shown in Fig. 2, using the RRRP linkage. The relative pole P_{23} lies always over the physical point B. Thus the velocity $\mathbf{u}_{23} = \mathbf{v}_B$, simplifying the problem and making direct the solution. But in a general situation, there is no physical point associated to the relative pole. Then, the convenience of developing an expression to calculate in an easy way the pole velocity of a relative pole arises.

The paper is organized as follows. In Section 2, the absolute motion of two bodies is studied and an equation relating the three absolute pole velocities is proposed. In Section 3, the formula is generalized by studying the relative motion of three bodies. In Section 4, the proposed formula is validated in two representative examples of planar linkages. Finally in Section 5, the vectorial formula is projected in its two components verifiying its coherence with the Hartman's Theorem and proposing the concept of centroma.

2. Studying the absolute motion of two bodies

Aronhold-Keneddy's theorem states that the three poles related to the motion between three rigid bodies lie permanently on the same line. Focusing on the case of two rigid bodies in motion with respect to a fixed one (frame), Fig. 3 shows the velocity fields of both moving rigid bodies along the line that joins the three poles.

$$\boldsymbol{v}_{A2} = \boldsymbol{\omega}_2 \times \mathbf{r}_{P_{21}P_{23}}; \quad tan\varphi_2 = \frac{v_{A2}}{r_{P_{21}P_{23}}} = \omega_2 \tag{2}$$

$$\boldsymbol{v}_{A3} = \boldsymbol{\omega}_3 \times \mathbf{r}_{P_{31}P_{23}}; \quad tan\varphi_3 = \frac{v_{A3}}{r_{P_{31}P_{23}}} = \omega_3 \tag{3}$$

$$\boldsymbol{v}_{A2} = \boldsymbol{v}_{A3} \rightarrow \boldsymbol{\omega}_2 \times \mathbf{r}_{P_{21}P_{23}} = \boldsymbol{\omega}_3 \times \mathbf{r}_{P_{31}P_{23}} \rightarrow \boldsymbol{\omega}_2 \times \mathbf{r}_{P_{21}P_{23}} = \boldsymbol{\omega}_3 \times \left(\mathbf{r}_{P_{21}P_{23}} - \mathbf{r}_{P_{21}\ P_{31}}\right) \tag{4}$$

From Fig. 3, the velocity of the relative pole P_{23}, $\boldsymbol{v}_{A2} = \boldsymbol{v}_{A3}$, can be expressed as follows.

Since the study is in planar kinematics and thus the angular velocities are normal to the moving plane, Eq. (4) can be expressed as,

$$\mathbf{r}_{P_{21}P_{23}} = \mathbf{r}_{P_{21}\ P_{31}} \cdot \frac{\omega_3}{\omega_3 - \omega_2} = \mathbf{r}_{P_{21}\ P_{31}} \cdot \frac{\omega_3}{\omega_{3r_2}} \tag{5}$$

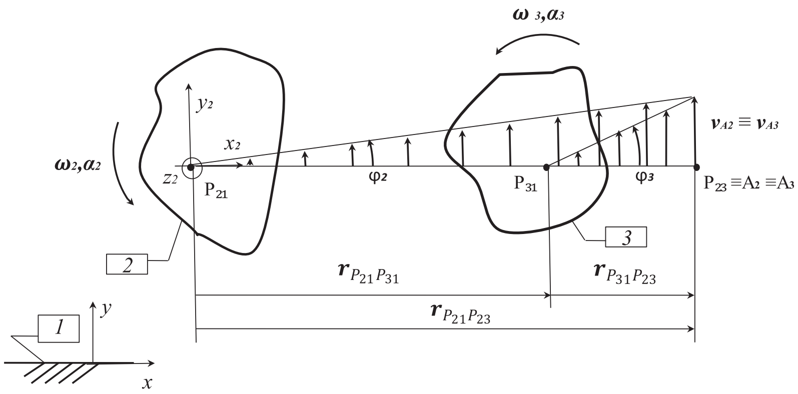

Fig. 3. Velocity fields of two moving bodies.

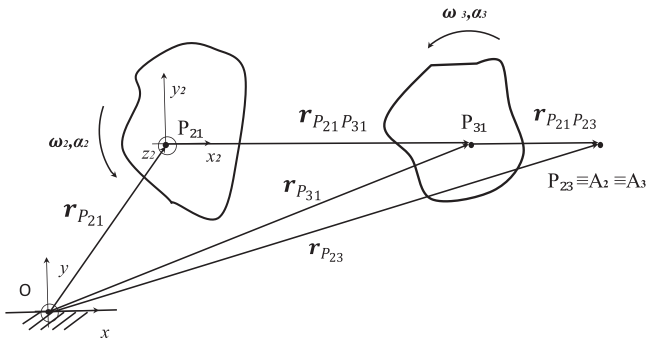

Fig. 4. Position vector diagram of two moving solids.

Eq. (5) is a relation between the distances of the poles. Deriving (5) with respect to the time,

$$\frac{d\left(\mathbf{r}_{P_{21}P_{23}}\right)}{dt} = \frac{d\left(\mathbf{r}_{P_{21}\ P_{31}}\right)}{dt}\frac{\omega_3}{\omega_3 - \omega_2} + \mathbf{r}_{P_{21}\ P_{31}}\frac{d}{dt}\left(\frac{\omega_3}{\omega_3 - \omega_2}\right) \tag{6}$$

From Fig. 4, it is possible to express the two vectors that appear in Eq. (6) as,

$$\mathbf{r}_{P_{21}P_{23}} = \mathbf{r}_{OP_{23}} - \mathbf{r}_{OP_{21}} \tag{7}$$

$$\mathbf{r}_{P_{21}P_{31}} = \mathbf{r}_{OP_{31}} - \mathbf{r}_{OP_{21}} \tag{8}$$

Then the derivatives of (7) and (8) can be expressed as follows,

$$\frac{d\left(\mathbf{r}_{P_{21}P_{23}}\right)}{dt} = \frac{d\left(\mathbf{r}_{OP_{23}}\right)}{dt} - \frac{d\left(\mathbf{r}_{OP_{21}}\right)}{dt}; \quad \frac{d\left(\mathbf{r}_{P_{21}P_{23}}\right)}{dt} = \boldsymbol{u}_{23} - \boldsymbol{u}_{21} \tag{9}$$

$$\frac{d\left(\mathbf{r}_{P_{21}P_{31}}\right)}{dt} = \frac{d\left(\mathbf{r}_{OP_{31}}\right)}{dt} - \frac{d\left(\mathbf{r}_{OP_{21}}\right)}{dt}; \quad \frac{d\left(\mathbf{r}_{P_{21}P_{31}}\right)}{dt} = \boldsymbol{u}_{31} - \boldsymbol{u}_{21} \tag{10}$$

As the position vectors refer to mathematical points, the velocity fields of the rigid bodies are not applicable. Also, the rule for derivation in moving bases cannot be used. So, velocities obtained are the absolute pole velocities.

$$\boldsymbol{u}_{23} - \boldsymbol{u}_{21} = \left(\boldsymbol{u}_{31} - \boldsymbol{u}_{21}\right)\frac{\omega_3}{\omega_3 - \omega_2} + \mathbf{r}_{P_{21}P_{31}}\frac{\omega_3\alpha_2 - \omega_2\alpha_3}{\left(\omega_3 - \omega_2\right)^2} \tag{11}$$

$$\boldsymbol{u}_{23} = \frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{21}\ \omega_2}{\omega_3 - \omega_2} + \mathbf{r}_{P_{21}P_{31}}\frac{\omega_3\alpha_2 - \omega_2\alpha_3}{\left(\omega_3 - \omega_2\right)^2} \tag{12}$$

Eq. (12) is the basis of the whole study that follows. It relates three absolute pole velocities and the position vector of the corresponding poles with angular velocities and accelerations of the moving bodies.

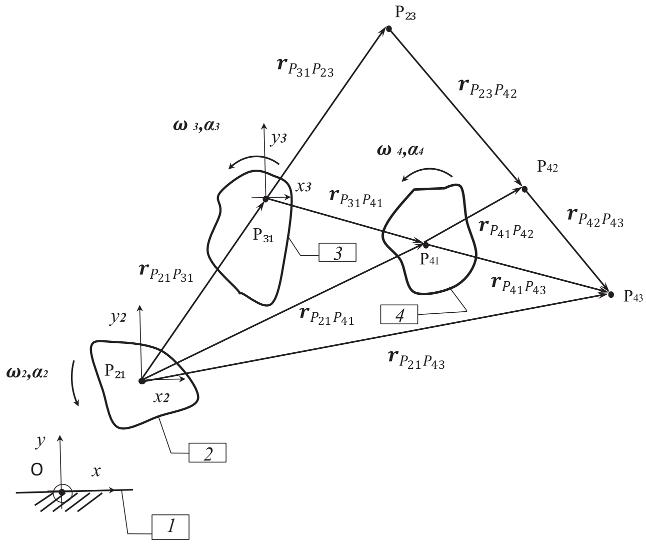

Fig. 5. Position vector relations between the three moving bodies.

3. Studying the relative motion of three bodies

In this section, Eq. (12) will be generalized for the case of three moving bodies 2, 3 and 4, that is $\boldsymbol{u}_{34} = f(\boldsymbol{u}_{23}, \boldsymbol{u}_{24})$. Applying Eq. (12) between the three bodies (Fig. 5),

$$\boldsymbol{u}_{24} = \frac{\boldsymbol{u}_{41}\omega_4 - \boldsymbol{u}_{21}\omega_2}{\omega_4 - \omega_2} + \mathbf{r}_{P_{21}P_{41}} \frac{\omega_4\alpha_2 - \omega_2\alpha_4}{(\omega_4 - \omega_2)^2} \tag{13}$$

$$\boldsymbol{u}_{23} = \frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{21}\omega_2}{\omega_3 - \omega_2} + \mathbf{r}_{P_{21}P_{31}} \frac{\omega_3\alpha_2 - \omega_2\alpha_3}{(\omega_3 - \omega_2)^2} \tag{14}$$

$$\boldsymbol{u}_{34} = \frac{\boldsymbol{u}_{41}\omega_4 - \boldsymbol{u}_{31}\omega_3}{\omega_4 - \omega_3} + \mathbf{r}_{P_{31}P_{41}} \frac{\omega_4\alpha_3 - \omega_3\alpha_4}{(\omega_4 - \omega_3)^2} \tag{15}$$

For the resolution of this system of three equations in three unknowns, the following parameters are defined. By using them, the solving is highly simplified since computer softwares like Mathematica cannot obtain directly the solution sought,

$$k_{ijk} = \frac{\omega_i}{\omega_j - \omega_k} \tag{16}$$

$$\boldsymbol{t}_{ij} = \boldsymbol{u}_{ij} - \mathbf{r}_{P_{i1}P_{j1}} \frac{\omega_j\alpha_i - \omega_i\alpha_j}{(\omega_j - \omega_i)^2} \tag{17}$$

$$\boldsymbol{t}_{24} = k_{442}\boldsymbol{u}_{41} - k_{242}\boldsymbol{u}_{21} \tag{18}$$

$$\boldsymbol{t}_{23} = k_{332}\boldsymbol{u}_{31} - k_{232}\boldsymbol{u}_{21} \tag{19}$$

$$\boldsymbol{t}_{34} = k_{443}\boldsymbol{u}_{41} - k_{343}\boldsymbol{u}_{31} \tag{20}$$

$$\begin{bmatrix} \boldsymbol{t}_{24} \\ \boldsymbol{t}_{23} \\ \boldsymbol{t}_{34} \end{bmatrix} = \begin{bmatrix} -k_{242} & 0 & k_{442} \\ -k_{232} & k_{332} & 0 \\ 0 & -k_{343} & k_{443} \end{bmatrix} . \begin{bmatrix} \boldsymbol{u}_{21} \\ \boldsymbol{u}_{31} \\ \boldsymbol{u}_{41} \end{bmatrix} \tag{21}$$

$$\boldsymbol{u}_{21} = \frac{\boldsymbol{t}_{24}\begin{vmatrix} k_{332} & 0 \\ -k_{343} & k_{443} \end{vmatrix} - \boldsymbol{t}_{23}\begin{vmatrix} 0 & k_{442} \\ -k_{343} & k_{443} \end{vmatrix} + \boldsymbol{t}_{34}\begin{vmatrix} 0 & k_{442} \\ k_{332} & 0 \end{vmatrix}}{\begin{vmatrix} -k_{242} & 0 & k_{442} \\ -k_{232} & k_{332} & 0 \\ 0 & -k_{343} & k_{443} \end{vmatrix}} \tag{22}$$

Having in mind the concepts of relative angular velocity and acceleration,

$$\boldsymbol{\alpha}_{3_{r_2}} = \boldsymbol{\alpha}_3 - \boldsymbol{\alpha}_2 - \boldsymbol{\omega}_4 \times \boldsymbol{\omega}_3 \tag{23}$$

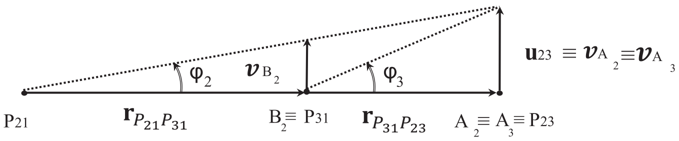

Fig. 6. Geometrical relations.

In the planar motion, the Resal complementary angular acceleration $-\boldsymbol{\omega}_4 \times \boldsymbol{\omega}_3$ turns out to be zero [12,13]. Thus, the equations are,

$$\omega_{3_{r_2}} = \omega_3 - \omega_2;\ \alpha_{3_{r_2}} = \alpha_3 - \alpha_2 \tag{24}$$

$$\omega_{4_{r_2}} = \omega_4 - \omega_2; \quad \alpha_{4_{r_2}} = \alpha_4 - \alpha_2 \tag{25}$$

From Fig. 5,

$$\mathbf{r}_{P_{31}P_{41}} = \mathbf{r}_{P_{21}P_{41}} - \mathbf{r}_{P_{21}P_{31}} \tag{26}$$

$$\mathbf{r}_{P_{23}P_{24}} = \mathbf{r}_{P_{21}P_{24}} - \mathbf{r}_{P_{21}P_{23}} \tag{27}$$

Developing the expression (22) by means of the Cramer's method and incorporating (24)–(27), the formula of $\boldsymbol{u}_{34}$ turns out to be of the following form,

$$\begin{aligned}\boldsymbol{u}_{34} = {} & \frac{\boldsymbol{u}_{24}\omega_{4_{r_2}} - \boldsymbol{u}_{23}\omega_{3_{r_2}}}{(\omega_{4_{r_2}} - \omega_{3_{r_2}})} + \frac{\mathbf{r}_{P_{21}P_{41}}}{(\omega_{4_{r_2}} - \omega_{3_{r_2}})^2}\left[\alpha_{3_{r_2}}\omega_4 - \alpha_{4_{r_2}}\omega_3 + \frac{\alpha_{4_{r_2}}\omega_2\left[\omega_{4_{r_2}} - \omega_{3_{r_2}}\right]}{\omega_{4_{r_2}}}\right] \\ & - \frac{\mathbf{r}_{P_{21}P_{31}}}{(\omega_{4_{r_2}} - \omega_{3_{r_2}})^2}\left[-\alpha_{4_{r_2}}\omega_3 + \alpha_{3_{r_2}}\omega_4 + \frac{\alpha_{3_{r_2}}\omega_2\left[\omega_{3_{r_2}} - \omega_{4_{r_2}}\right]}{\omega_{3_{r_2}}}\right]\end{aligned} \tag{28}$$

Considering Fig. 2 and taking into account the geometrical relation shown in Fig. 6,

$$\omega_2 = \tan\varphi_2 = \frac{v_{B_2}}{\mathrm{r}_{P_{21}P_{31}}} \tag{29}$$

$$\omega_3 = tan\ \varphi_3 = \frac{v_{A_3}}{\mathrm{r}_{P_{31}P_{23}}} \tag{30}$$

Substituting these relations together with (2), (3) and (5) and grouping terms, the final compact formula (31) is obtained:

$$\boldsymbol{u}_{34} = \frac{\boldsymbol{u}_{24}\omega_{4_{r_2}} - \boldsymbol{u}_{23}\omega_{3_{r_2}}}{(\omega_{4_{r_2}} - \omega_{3_{r_2}})} + \mathbf{r}_{P_{23}P_{24}}\frac{\alpha_{3_{r_2}}\omega_{4_{r_2}} - \alpha_{4_{r_2}}\omega_{3_{r_2}}}{(\omega_{4_{r_2}} - \omega_{3_{r_2}})^2} \tag{31}$$

This generalization leads to the same expression as (12). There are no additional terms like those derived from Resal and Coriolis accelerations, since the formula is still in the velocity level.

It is important to remark that this is not an obvious result. Expressing Eq. (12) with respect to a mobile observer, for example body 2, is indeed somewhat trivial. That would be the case of a formula that relates relative poles velocities as shown in (32).

$$\boldsymbol{u}_{34}{}^{②} = f_1\left(\boldsymbol{u}_{31}{}^{②}, \boldsymbol{u}_{41}{}^{②}\right) \tag{32}$$

The case studied in this section is different. In fact, the formula obtained relates absolute pole velocities.

$$\boldsymbol{u}_{34} = f_2(\boldsymbol{u}_{32}, \quad \boldsymbol{u}_{42}) \tag{33}$$

4. Illustrative examples

In this section two illustrative examples of application are presented.

The first example is the case of a 1-dof guiding mechanism composed of two four-bar linkages synchronized by a coupler bar (Fig. 7). The goal is to obtain the absolute pole velocity $\boldsymbol{u}_{41}$.

In this case, the relative poles permanently lie on the mechanism revolute pairs. Table 1 shows the data provided by the GIM software [3] after a kinematic analysis, for the position depicted in Fig. 7 and an input of $\omega_2 = 2\Pi\mathrm{rad/s}$ and $\alpha_2 = 0\,\mathrm{rad/s}^2$.

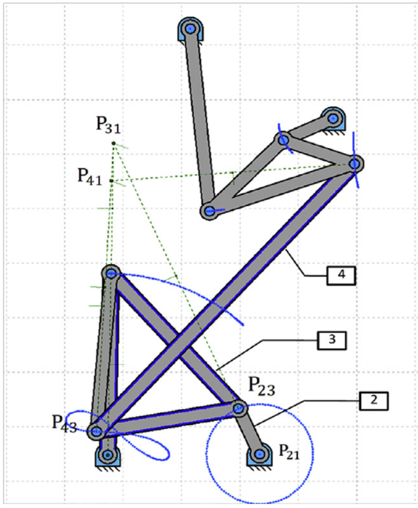

Fig. 7. Application example 1.

Table 1
Angular velocities and accelerations.

Elements	2	3	4
Angular velocity: $\boldsymbol{\omega}$ (rad/s)	2Π	−1.07	−1.24
Angular acceleration: $\boldsymbol{\alpha}$ (rad/$\mathbf{s}^2$)	0	−15.11	−0.62

Table 2
Poles and pole velocities.

Velocity Poles	Horizontal Component of the Velocity Poles (m)	Vertical component of the Velocity Poles (m)	Pole velocities	Horizontal Component of the Pole Velocities (m/s)	Vertical Component of the Pole Velocities (m/s)
P_{21}	0.33	0.34	$\boldsymbol{u}_{21}$	0	0
P_{23}	0.27	0.60	$\boldsymbol{u}_{23}$	−0.52	−0.23
P_{43}	0.05	0.39	$\boldsymbol{u}_{43}$	−0.57	0.03
P_{31}	0.08	0.91	$\boldsymbol{u}_{31}$	−0.58	−8.38
P_{41}	0.07	0.85	$\boldsymbol{u}_{41}$	−0.24	−1.08

Eq. (12) is first applied to links 2 and 3 to obtain $\boldsymbol{u}_{31}$. Then, it is applied to 3 and 4 to achieve $\boldsymbol{u}_{41}$.

$$\boldsymbol{u}_{23} = \frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{21}\omega_2}{\omega_3 - \omega_2} + \mathbf{r}_{P_{21}P_{31}} \frac{\omega_3\alpha_2 - \omega_2\alpha_3}{(\omega_3 - \omega_2)^2} \rightarrow \boldsymbol{u}_{31} \tag{34}$$

$$\boldsymbol{u}_{43} = \frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{41}\omega_4}{\omega_3 - \omega_4} + \mathbf{r}_{P_{41}P_{31}} \frac{\omega_3\alpha_4 - \omega_4\alpha_3}{(\omega_3 - \omega_4)^2} \rightarrow \boldsymbol{u}_{41} \tag{35}$$

Being the result of $\boldsymbol{u}_{31}$ and $\boldsymbol{u}_{41}$ (Table 2) coincident with those obtained from the kinematic analysis made using GIM software.

The second example is the case of another 1-dof guiding mechanism composed in this case by one four-bar linkage and a slider-crank mechanism linked by a coupler bar (Fig. 8). The goal of the problem is to obtain the absolute pole velocity $\boldsymbol{u}_{41}$.

Again, the relative poles permanently lie on the mechanism revolute pairs. Table 3 shows the data provided by the GIM software [3] for the position of Fig. 8 and an input of $\omega_2 = 2\Pi$ [rad/s] and $\alpha_2 = 0$ [rad/s^2].

Eq. (12) is first applied to links 2 and 3 to obtain $\boldsymbol{u}_{31}$. Then, it is applied to 3 and 4 to achieve $\boldsymbol{u}_{41}$.

$$\boldsymbol{u}_{23} = \frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{21}\omega_2}{\omega_3 - \omega_2} + \mathbf{r}_{P_{21}P_{31}} \frac{\omega_3\alpha_2 - \omega_2\alpha_3}{(\omega_3 - \omega_2)^2} \rightarrow \boldsymbol{u}_{31} \tag{36}$$

$$\boldsymbol{u}_{43} = \frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{41}\omega_4}{\omega_3 - \omega_4} + \mathbf{r}_{P_{41}P_{31}} \frac{\omega_3\alpha_4 - \omega_4\alpha_3}{(\omega_3 - \omega_4)^2} \rightarrow \boldsymbol{u}_{41} \tag{37}$$

Being the result of $\boldsymbol{u}_{31}$ and $\boldsymbol{u}_{41}$ the same as the kinetic analysis provided by the GIM program as shown in Table 4.

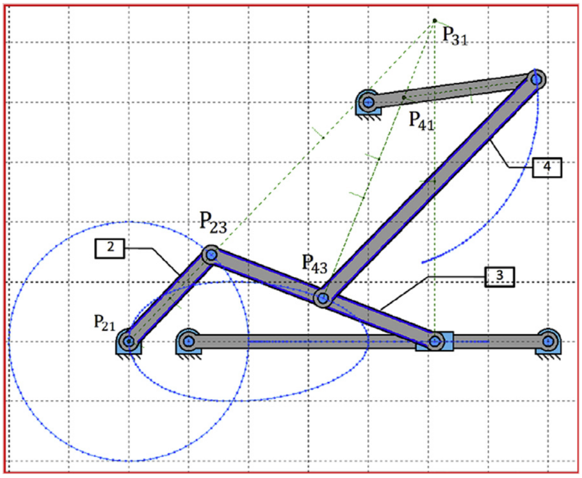

Fig. 8. Application example 2.

Table 3
Angular velocities and accelerations.

Elements	2	3	4
Angular velocity: $\boldsymbol{\omega}$ (rad/s)	2Π	−2.32	−3.21
Angular acceleration: $\boldsymbol{\alpha}$ (rad/$\mathbf{s}^2$)	0	13.24	−25.95

Table 4
Poles and pole velocities.

Velocity Poles	Horizontal Component of the Velocity Poles (m)	Vertical component of the Velocity Poles (m)	Pole Velocities	Horizontal Component of the Pole Velocities (m/s)	Vertical Component of the Pole Velocities (m/s)
P_{21}	0.20	0.40	$\boldsymbol{u}_{21}$	0	0
P_{23}	0.33	0.54	$\boldsymbol{u}_{23}$	−0.91	0.86
P_{43}	0.52	0.47	$\boldsymbol{u}_{43}$	−1.07	0.43
P_{31}	0.71	0.93	$\boldsymbol{u}_{31}$	−1.24	5.44
P_{41}	0.65	0.80	$\boldsymbol{u}_{41}$	−3.05	−0.56

5. Projecting the formula in its two components

As said, Eq. (12) is a vectorial relation. So, it can be projected on specific axes to obtain two algebraic equations.

In Fig. 9, is depicted the graphical construction derived from Hartman's theorem [5]. The moving plane 2 corresponds to the one in study of known movement. The moving plane 3 will be the so called "normal plane", which is the one associated to the motion of the tangent line to the path of point A.

5.1. Projection along the tangential direction

Hartman's theorem states that the tip of the velocity vector of a moving point, the center of curvature of its trajectory, and the component of the pole velocity parallel to the point velocity vector, are aligned (Fig. 9). Thus,

$$\frac{u_{23}}{u'_{21}} = \frac{\mathrm{r}_{P_{31}P_{23}}}{\mathrm{r}_{P_{21}P_{31}}} \tag{38}$$

This is a scalar proportion that can also be verified by the formula proposed in this article. To obtain the same relation, formula (12) is projected along the tangent line of the trajectory of A. In Eq. (39) both terms of Eq. (12) are premultiplied by the unit vector $\boldsymbol{t}$.

$$\boldsymbol{t}\cdot\boldsymbol{u}_{23} = \boldsymbol{t}\cdot\frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{21}\omega_2}{\omega_3-\omega_2} + \boldsymbol{t}\ \cdot\mathbf{r}_{P_{21}P_{31}}\frac{\omega_3\alpha_2-\omega_2\alpha_3}{(\omega_3-\omega_2)^2} \tag{39}$$

$$\boldsymbol{t}\cdot\ \boldsymbol{u}_{23} = \boldsymbol{t}\cdot\boldsymbol{u}_{21}\frac{-\omega_2}{\omega_3-\omega_2} \rightarrow \frac{u_{23}}{u'_{21}} = \frac{\omega_2}{\omega_3-\omega_2} \tag{40}$$

And using Eq. (5),

$$\mathrm{r}_{P_{21}P_{23}} = \mathrm{r}_{P_{21}P_{31}}\cdot\frac{\omega_3}{\omega_3-\omega_2};\quad \mathrm{r}_{P_{31}P_{23}} = \mathrm{r}_{P_{21}P_{23}} - \mathrm{r}_{P_{21}P_{31}} \tag{41}$$

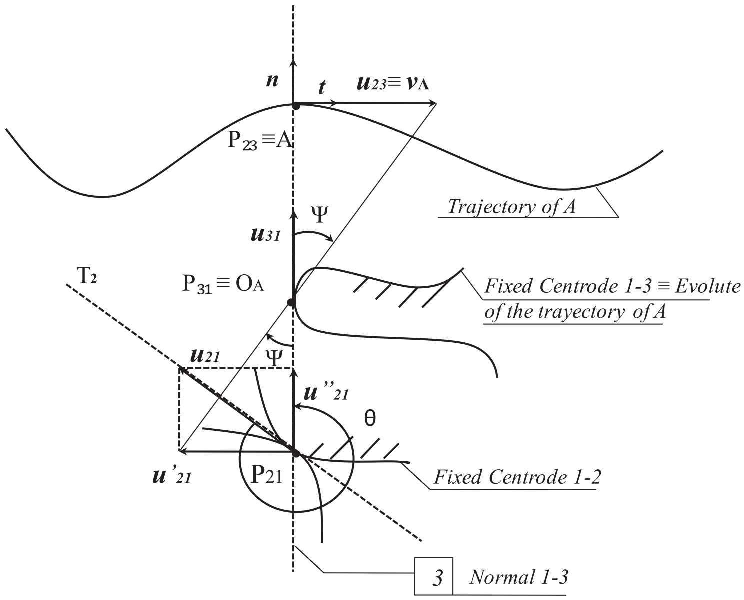

Fig. 9. Graphical construction of Hartman's Theorem.

$$\mathrm{r}_{P_{31}P_{23}} = \mathrm{r}_{P_{21}P_{31}} \cdot \left(\frac{\omega_3}{\omega_3 - \omega_2} - \frac{\omega_3 - \omega_2}{\omega_3 - \omega_2} \right) \rightarrow \frac{\mathrm{r}_{P_{31}P_{23}}}{\mathrm{r}_{P_{21}P_{31}}} = \frac{\omega_2}{\omega_3 - \omega_2} \tag{42}$$

Combining (40) and (42), it is finally obtained the same proportion as the one described in Hartman's theorem.

$$\frac{u_{23}}{u'_{21}} = \frac{\mathrm{r}_{P_{31}P_{23}}}{\mathrm{r}_{P_{21}P_{31}}} \tag{43}$$

5.2. Projection along the normal direction.Velocity of the mathematical point associated with the centre of curvature

As it is known, the centre of curvature of the path of a point is a fixed point. Thus, a mathematical point O_A will be defined in order to study the position variation of the different centres of curvature of the path generated by A. In this paper is used the name *centroma* to refer to this concept. Consequently the path described by this mathematical point O_A will be the evolute curve of the path generated by the motion of A.

Now the objective is to obtain an expression that provides the velocity module of such mathematical point associated to the center of curvature of the path of any mobile point. As expressed in Fig. 7, from Hartman's theorem:

$$\frac{\mathrm{r}_{O_A A}}{v_A} = \frac{\mathrm{r}_{O_A P_{21}}}{u_{21}{}'} \tag{44}$$

Simplifying,

$$\mathrm{r}_{O_A P_{21}} = \frac{1}{\frac{1}{\delta_2 sen\theta} - \frac{1}{\mathrm{r}_{P_{21}A}}} \tag{45}$$

Being δ_2 the diameter of the inflection circle, and θ angle formed by the position vector of point A in the Euler-Savary local reference frame. Additionally, from Fig. 9 can also be obtained obtain the angular velocity of the rigid body 3.

$$\omega_3 \mathrm{r}_{O_A P_{21}} = u_{21}{}' \tag{46}$$

$$\omega_3 = \frac{u_{21}{}'}{\mathrm{r}_{O_A P_{21}}} \rightarrow \omega_3 = \frac{\omega_2 \delta_2 sen\theta}{\mathrm{r}_{O_A P_{21}}} \rightarrow \omega_3 = \omega_2 \delta_2 sen\theta \left(\frac{1}{\delta_2 \mathrm{sen}\theta} - \frac{1}{\mathrm{r}_{P_{21}A}} \right) \tag{47}$$

$$\omega_3 = \omega_2 \left(1 - \frac{\delta_2}{\mathrm{r}_{P_{21}A}} sen\theta \right) \tag{48}$$

Being,

$$\gamma = \frac{\delta_2}{\mathrm{r}_{P_{21}A}} sen\theta \tag{49}$$

$$\omega_3 = \omega_2 (1 - \lambda) \tag{50}$$

This angular velocity w_3 represents the rotation of the normal line to the path. Deriving this expression with respect to the time,

$$\alpha_3 = \alpha_2(1-\gamma) - \omega_2 \frac{d\gamma}{dt} \tag{51}$$

As explained, Eq. (12) has a vectorial form. In this section, the equation is projected in the normal direction **n** obtaining,

$$\boldsymbol{n} \cdot \boldsymbol{u}_{23} = \boldsymbol{n} \cdot \frac{\boldsymbol{u}_{31}\omega_3 - \boldsymbol{u}_{21}\ \omega_2}{\omega_3 - \omega_2} + \boldsymbol{n} \cdot \mathbf{r}_{P_{21}P_{31}} \frac{\omega_3\alpha_2 - \omega_2\alpha_3}{(\omega_3 - \omega_2)^2} \tag{52}$$

Being,

$$\boldsymbol{n} \cdot \boldsymbol{u}_{23} = 0 \tag{53}$$

$$\boldsymbol{n} \cdot \boldsymbol{u}_{21} = u_{21}{''} \tag{54}$$

$$\mathbf{r}_{P_{21}P_{31}} = -\mathbf{r}_{O_AP_{21}} \tag{55}$$

$$\boldsymbol{n} \cdot \mathbf{r}_{O_AP_{21}} = -\mathrm{r}_{O_AP_{21}} \tag{56}$$

Substituting (53), (54), (55) and (56) in (52),

$$u_{31}\omega_3 = u_{21}{''}\ \omega_2 - \mathrm{r}_{O_AP_{21}} \frac{\omega_3\alpha_2 - \omega_2\alpha_3}{\omega_3 - \omega_2} \tag{57}$$

Substituting the previous results of angular velocity and accelerations (50), (51) in (57),

$$u_{O_A}\omega_2(1-\gamma) = u_{21}{''}\ \omega_2 - \mathrm{r}_{O_AP_{21}} \frac{\omega_2\alpha_2(1-\gamma) - \omega_2\alpha_2(1-\gamma) + \omega_2{}^2\frac{d\gamma}{dt}}{\omega_2(1-\gamma) - \omega_2} \tag{58}$$

$$u_{O_A}(1-\gamma) = u_{21}\cos\theta + \mathrm{r}_{O_AP_{21}} \frac{1}{\gamma}\frac{d\gamma}{dt} \tag{59}$$

So, the velocity of the centroma, has its direction normal to the trajectory of the point, and being its module,

$$u_{O_A} = u_{21}\frac{\cos\theta}{(1-\gamma)} + \frac{\mathrm{r}_{O_AP_{21}}}{\gamma(1-\gamma)}\frac{d\gamma}{dt} \tag{60}$$

where,

$$\gamma = \frac{\delta_2}{\mathrm{r}_{P_{21}A}} sen\theta \tag{61}$$

From Eq. (60) it can be observed that exists a locus of centromas, whose speed is infinite at a certain instant,

$$1 - \gamma = 0 \tag{62}$$

$$\delta_2 sen\theta = \mathrm{r}_{P_{21}A} \tag{63}$$

It can be seen, how this locus corresponds to the inflection circle.

The inflection circle is the locus of physical points whose normal acceleration is instantaneously zero. Thus, they describe a linear path during a differential step of time. In other words, the center of curvature of the path described by these points lies at infinity.

As explained, when substituting the locus (63) in the Eq. (45) the result turns out to be an infinite distance between the pole and the centroma, verifying the feature of the inflection circle.

$$\mathrm{r}_{O_AP_{21}} = \frac{1}{\frac{1}{\delta_2 sen\theta} - \frac{1}{\delta_2 sen\theta}} = \frac{1}{0} \tag{64}$$

Hence according to this result, the centromas at infinity have an infinite speed.

6. Conclusions

In this paper by studying the relative planar motion of three bodies, has been presented a formula for the calculation of the absolute pole velocities of the relative poles in the planar motion of three rigid bodies. It relates three absolute pole velocities and the position vector of the corresponding poles with the angular velocities and accelerations of the moving bodies. The formula allows the calculation of one of the absolute pole velocities from the other two, being known the angular velocities and accelerations of the moving bodies. It is applicable regardless of whether the instantaneous centers (poles) are located on physical points on the linkage or not. Previously, in Section 2 the formula for the case of the absolute motion of two moving bodies has been obtained. It is verified that this is a particularization of the proposed formula.

The formula is applied to two representative guiding mechanisms: the first one composed of two four-bar linkages synchronized by a coupler and the second one composed by one four-bar linkage and a slider-crank mechanism linked by a coupler bar All the examples analysed and the calculations done, have been simulated and verified using the GIM research and educational numerical software.

Finally, the vectorial formula is projected in its two components verifiying its coherence with the Hartman's Theorem and proposing the concept of *centroma.* This concept corresponds to a mathematical point associated to the centers of curvature of the path described by a physical point of the moving body. Finally, the velocity of the *centroma* is obtained.

The future work is focused on the application of the velocity of the *centroma* to obtain the locus of points in a moving solid that traces circumferential paths.

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